

Energetics of interacting magnetized domains

David P. Jackson* and Brennan Gantner

Department of Physics, Santa Clara University, Santa Clara, California 95053

(Received 21 June 2001; published 26 October 2001)

Many of the pattern forming features of ferrofluids, lipid monolayers, type-I superconductors, and magnetic bubbles can be understood by treating them as dipolar (uniformly magnetized or polarized) domains. Here, we investigate the early stages of pattern formation in a system consisting of two quasi-two-dimensional dipolar domains. We calculate the linearized interaction energy for these domains and find that the lowest energy states are those for which each drop has an indentation directed toward the other. These theoretically preferred states compare favorably with results of experiments performed with ferrofluids.

DOI: 10.1103/PhysRevE.64.056230

PACS number(s): 75.50.Mm, 68.18.-g, 75.70.Kw, 47.54.+r

I. INTRODUCTION

Dipolar domain energetics [1] has been used as a model for the pattern formation process in a wide range of physical systems including Langmuir films [2,3], magnetic bubbles [4], type-I superconductors [5,6], and ferrofluids [7]. Using this and other formalisms, a single domain has been extensively studied [8–10] and this has led to a good understanding of the pattern selection mechanism. A hydrodynamic approach has also proved useful in this context [11] and the resulting patterns are virtually indistinguishable from those obtained using the energetic approach. These studies have focused primarily on the fingering instability for a single domain but have not investigated how the presence of multiple domains might affect the patterns that form. Those that have considered multiple domains [12] have used numerical simulations to focus on the characteristic distance between stripes in a highly disordered array. Here, we seek to find the lowest energy configurations for a system of two domains in the earliest stages of the pattern formation process. These configurations are extremely important because it is from these early stages that the final state patterns emerge. In some sense, the entire structure of the final pattern becomes “frozen in” during these early times.

To begin, let us describe one particular physical system. Consider a ferrofluid drop of volume V trapped between two closely spaced glass plates separated by a distance h (a Hele-Shaw cell). In this geometry, the motion is essentially two dimensional and the boundary of the drop looks like a thin ribbon. We assume throughout this paper that the interface of the fluid is perpendicular to the glass plates and therefore focus our attention on the shape of the cross section. Incompressibility of the fluid keeps the cross-sectional area fixed even though the shape can change. When no magnetic field is applied, the shape of the drop is determined solely by surface tension and will take the form of a circle of radius R_0 . If a magnetic field is applied perpendicular to the glass plates, the ferrofluid becomes magnetized. This produces an outward magnetic force that competes with surface tension and results in the domain evolving into a branched pattern.

We assume that the magnetization M is uniform throughout the domain and oriented perpendicular to the Hele-Shaw cell.

A point on the interface of the domain can be described by a vector \mathbf{r} in the xy plane, and a point z . The vector \mathbf{r} gives the location along the length of the ribbon and z gives the height. After integrating over the height of the ribbon, the magnetic energy of an arbitrarily shaped simply connected domain can be written [1]

$$\mathcal{E}_{\text{mag}} = 2\pi M^2 V - M^2 h \oint ds \oint ds' \hat{\mathbf{t}} \cdot \hat{\mathbf{t}}' \Phi(R/h), \quad (1)$$

where $R = |\mathbf{r}(s) - \mathbf{r}(s')|$ is the *in-plane* distance between two points on the contour, $\hat{\mathbf{t}}$ and $\hat{\mathbf{t}}'$ are tangent vectors at these two points, and

$$\Phi(\xi) = \sinh^{-1}(1/\xi) + \xi - \sqrt{1 + \xi^2} \quad (2)$$

is a potential function that gives the strength of the interaction between points on the contour. Equation (1) is an exact expression for the energy of a simply connected “cylindrical-type” domain of arbitrary cross section (valid for any height h). The first term is proportional to the volume of the domain and is constant. The second term can be understood as a current-current interaction with coupling strength Φ . Since a uniformly magnetized domain is equivalent to a current ribbon flowing around the boundary of a nonmagnetized domain, this term follows by integrating the free self-energy of a current ribbon over its height.

To extend this to multidomain systems, we note that each domain will contribute an energy as in Eq. (1) but there will also be interaction terms of the form

$$\mathcal{E}_{\text{int}} = -M^2 h \oint ds_i \oint ds_j \hat{\mathbf{t}}_i \cdot \hat{\mathbf{t}}_j \Phi(R_{ij}/h), \quad (3)$$

where the subscripts refer to separate contours and $R_{ij} = |\mathbf{r}(s_i) - \mathbf{r}(s_j)|$ is the distance between a point on the i th contour and a point on the j th contour. To simplify our work, we make note of the following. First, the constant term in Eq. (1) will have no bearing on the pattern formation process at all. Second, the self-energy term from a particular domain will not affect the evolution of any other domain. Since our interest lies in how the interactions *between* domains will affect their evolution, we focus our attention on this interac-

*Present address: Dept. of Physics and Astronomy, Dickinson College, Carlisle, PA 17013.

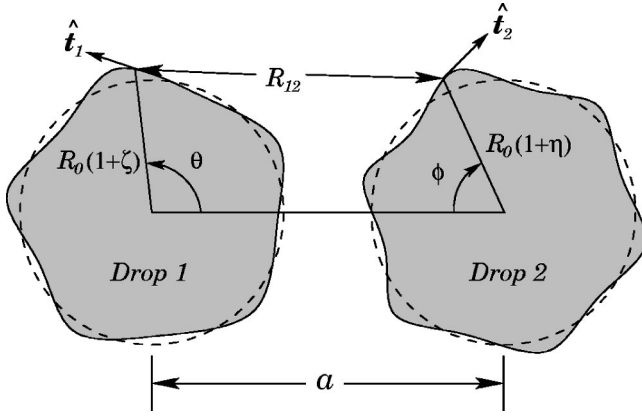


FIG. 1. Schematic representation of two initially circular ferrofluid drops of radius R_0 (dashed curves) with arbitrary perturbations (solid curve). The angles θ and ϕ are used to parametrize the curves and are measured with respect to the line joining the centers of the two drops as shown.

tion energy. In particular, we will use the interaction energy to calculate the preferred rotational position for nearly circular domains.

II. INTERACTIONS BETWEEN TWO DOMAINS

To determine the effect of interactions between multiple domains, we begin by examining the interaction energy of the two-domain system shown in Fig. 1. Both domains have an initial radius of R_0 and their centers are separated by a distance $a > 2R_0$. In an applied magnetic field, these two domains will acquire the same magnetization, or, equivalently, will have the same current (magnitude and direction) flowing around their boundaries. Notice, however, that the angles θ and ϕ in Fig. 1 are defined in such a way as to traverse the two contours in *opposite* directions. This is done to simplify our later results, but it is important to realize that this introduces a minus sign into Eq. (3) that must be taken into account when calculating the correct energy for our physical system. On each surface, we add an arbitrary perturbation such that $r_1(\theta) = R_0[1 + \zeta(\theta)]$ and $r_2(\phi) = R_0[1 + \eta(\phi)]$. It is assumed that these perturbations are small in the sense that $\zeta \ll 1$ and $\eta \ll 1$, and that their derivatives $\zeta' = d\zeta/d\theta$ and $\eta' = d\eta/d\phi$ are small as well. Our goal is to expand the interaction energy to first order in terms of these small parameters. We do not need to include the effects of surface tension because the surface energy is constant to first order. Moreover, the effects of surface tension are local to a specific drop and therefore play no role in the interactions between drops.

The first step is to linearize the elements in Eq. (3). This gives

$$ds_1 \approx R_0(1 + \zeta)d\theta, \quad (4a)$$

$$ds_2 \approx R_0(1 + \eta)d\phi, \quad (4b)$$

$$\hat{\mathbf{t}}_1 \cdot \hat{\mathbf{t}}_2 \approx \cos(\theta + \phi) + (\zeta' + \eta')\sin(\theta + \phi), \quad (4c)$$

$$\Phi(R_{12}/h) \approx \Phi(\Delta) + (\Delta - \sqrt{1 + \Delta^2})\epsilon, \quad (4d)$$

where we have defined

$$\Delta^2 = \left(\frac{p}{q}\right)^2 \left[1 + q^2 \cos^2\left(\frac{\theta + \phi}{2}\right) - q(\cos\theta + \cos\phi) \right], \quad (5)$$

$$\epsilon = \frac{p^2}{2q^2\Delta^2} \left[(\zeta + \eta)q^2 \cos^2\left(\frac{\theta + \phi}{2}\right) - q(\zeta \cos\theta + \eta \cos\phi) \right], \quad (6)$$

and have introduced two dimensionless parameters, the aspect ratio $p = 2R_0/h$ and the separation constant $q = 2R_0/a$. We focus our attention on the (dimensionless) interaction integral for the two-drop system, defined by

$$I_{\text{int}} = \frac{1}{R_0^2} \oint ds_1 \oint ds_2 \hat{\mathbf{t}}_1 \cdot \hat{\mathbf{t}}_2 \Phi(R_{12}/h). \quad (7)$$

Using the approximations given in Eqs. (4), we calculate the linearized interaction integral to be

$$I_{\text{int}} \approx I_{\text{int}}^{(0)} + \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \{ [(\zeta + \eta)\cos(\theta + \phi) + (\zeta' + \eta')\sin(\theta + \phi)] \Phi(\Delta) + (\Delta - \sqrt{1 + \Delta^2})\epsilon \cos(\theta + \phi) \}, \quad (8)$$

where

$$I_{\text{int}}^{(0)} = \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \cos(\theta + \phi) \Phi(\Delta) \quad (9)$$

is the interaction integral for two perfectly circular domains. Unfortunately, the complexity of the integrand precludes our finding a closed form expression even for the relatively simple case of circular domains.

To proceed further, we need to impose specific perturbations. We choose pure mode disturbances of the form $\zeta(\theta) = \zeta_n \cos[n(\theta + \alpha)]$ and $\eta(\phi) = \eta_m \cos[m(\phi + \beta)]$. Here, ζ_n and η_m are positive numbers that give the amplitude of the perturbations, n and m are mode numbers that specify the number of bumps in the perturbation, and α and β are rotation angles for the entire disturbance (see Fig. 2). Our goal is to minimize the interaction energy with respect to these rotation angles.

Inserting these pure mode disturbances into Eq. (8) and making use of the fact that the interaction energy is invariant under a direction change of both rotation angles (i.e., $\alpha, \beta \rightarrow -\alpha, -\beta$) leads to

$$I_{\text{int}} = I_{\text{int}}^{(0)} + \zeta_n A_n \cos n\alpha + \eta_m A_m \cos m\alpha, \quad (10)$$

with amplitude coefficients

$$A_n = \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \{ [\Phi(\Delta) + \Psi(\Delta)] \cos(\theta + \phi) \cos n\theta - n\Phi(\Delta) \sin(\theta + \phi) \sin n\theta \}. \quad (11)$$

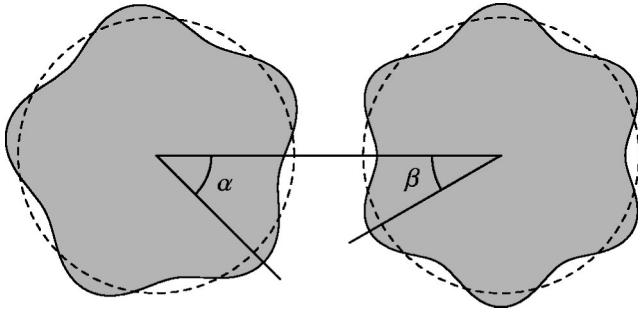


FIG. 2. Perturbations of the form $\zeta_n \cos[n(\theta + \alpha)]$ and $\eta_m \cos[m(\phi + \beta)]$ are supplied to initially circular domains (dashed curves). The mode numbers n and m specify the number of bumps while the angles α and β specify the amount of rotation. In this case, $n = 5$, $m = 6$, $\alpha = \pi/4$, and $\beta = \pi/6$.

Here, Φ and Δ are given by Eqs. (2) and (5) while

$$\Psi(\Delta) = \frac{p^2}{2q^2\Delta^2} \left[q^2 \cos^2 \left(\frac{\theta + \phi}{2} \right) - q \cos \theta \right] (\Delta - \sqrt{1 + \Delta^2}). \quad (12)$$

III. PREFERRED ORIENTATIONS

Equation (10) is our main result and has a surprisingly simple form. To find the preferred orientation of each drop, we simply need to find the minimum values for the interaction energy. Apart from a non-negative multiplicative constant, this interaction energy is

$$E_{\text{int}} = I_{\text{int}}^{(0)} + \zeta_n A_n \cos n\alpha + \eta_m A_m \cos m\beta. \quad (13)$$

Minimizing this energy with respect to the rotation angles α and β will then give the preferred orientations. Differentiating Eq. (13) yields extreme values at angles $\alpha^* = i\pi/n$ and $\beta^* = j\pi/m$, where i and j are integers. To find the energy minima, we require

$$\left. \frac{\partial^2 E_{\text{int}}}{\partial \alpha^2} \right|_{\alpha^*} = -n^2 \zeta_n A_n (-1)^i > 0 \quad (14a)$$

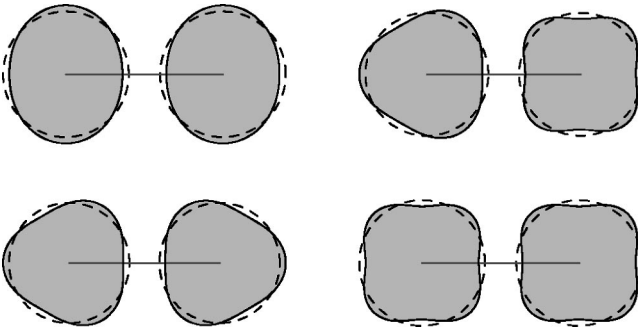


FIG. 3. The minimum energy configurations for four sets of domains with different perturbations. The dashed curves show an unperturbed circle. Notice that in each case the orientations are such that an indentation points directly along the line joining the two domains. This is a general feature that applies regardless of the mode number.

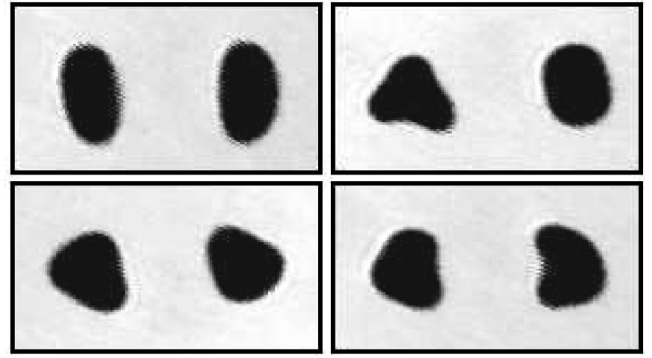


FIG. 4. Videotape still frames of ferrofluid drops taken a short time after a magnetic field has been applied. Although the perturbations do not look exactly like pure mode disturbances, there is a clear tendency for the drops to orient themselves with indentations directed along the line joining the drops. These configurations are quite reproducible as long as the drops are fairly close together.

and

$$\left. \frac{\partial^2 E_{\text{int}}}{\partial \beta^2} \right|_{\beta^*} = -m^2 \eta_m A_m (-1)^j > 0. \quad (14b)$$

Since ζ_n and η_m are positive by construction, solving Eqs. (14) depends critically on the amplitude coefficients A_n and A_m . Direct computation reveals that these coefficients are positive, which constrains i and j to be odd integers. Notice further that, because of the symmetry of our pure mode disturbances, these angles are degenerate in the sense that they all lead to the same physical state. Therefore, without loss of generality, we can focus our attention on a single rotation angle for each drop. The simplest case is $i = j = 1$, for which we obtain preferred angles

$$\alpha_0 = \frac{\pi}{n} \quad \text{and} \quad \beta_0 = \frac{\pi}{m}. \quad (15)$$

Figure 3 shows the preferred orientations for four pairs of nearly circular domains. As is evident in this figure, these minimum energy states are oriented such that each drop has an indentation directed toward the other. This makes good qualitative sense if you recall that these magnetic domains can be described as current loops. These currents are anti-aligned where the drops are closest together and this results in a repulsive force. Thus, the system can lower its interaction energy by increasing this distance. It is reassuring to note that this indentation alignment can be seen in the numerical simulations of Drikis *et al.* (Ref. [12]). Although they examined a more complex arrangement of a disordered set of drops, there is a clear tendency for indentation alignment to occur between neighboring drops. There are some cases, however, where this alignment is not observed. This appears to happen when the number of indentations is incommensurate with the number of neighboring drops. It would be interesting to examine analytically how these competing factors affect the pattern formation process.

As an example of how these interactions take place in a real physical system, Fig. 4 shows four pairs of ferrofluid drops a short time after a magnetic field was applied. The similarity to Fig. 3 is clearly apparent. Interestingly, these configurations were extremely reproducible as long as the drops were fairly close together initially. Not surprisingly, when the drops were more than a few diameters apart, the interactions had little effect on the pattern formation and this reproducibility was lost.

ACKNOWLEDGMENTS

We are grateful to the physics faculty at Dickinson College for making available the equipment and laboratory space to perform the experiments, and thank Matthew Welker and Russell LaMantia for helping put the laboratory together. This work was supported in part by Research Corporation Grant No. CC4693 and a Santa Clara/IBM Faculty Research Grant.

-
- [1] S.A. Langer, R.E. Goldstein, and D.P. Jackson, *Phys. Rev. A* **46**, 4894 (1992).
 - [2] K.Y.C. Lee and H.M. McConnell, *J. Phys. Chem.* **97**, 9532 (1993).
 - [3] R.E. Goldstein and D.P. Jackson, *J. Phys. Chem.* **98**, 9626 (1994).
 - [4] M. Seul and D. Andelman, *Science* **267**, 476 (1995).
 - [5] R.P. Huebener, *Magnetic Flux Structures in Superconductors* (Springer-Verlag, New York, 1979).
 - [6] R.E. Goldstein, D.P. Jackson, and A.T. Dorsey, *Phys. Rev. Lett.* **76**, 3818 (1996).
 - [7] R.E. Rosensweig, *Ferrohydrodynamics* (Cambridge University Press, Cambridge, 1985).
 - [8] A.O. Cebers and M.M. Mairov, *Magnetohydrodynamics* (N.Y.) **16**, 21 (1980).
 - [9] A.J. Dickstein *et al.*, *Science* **261**, 1012 (1993).
 - [10] J.A. Miranda and M. Widom, *Phys. Rev. E* **55**, 3758 (1997).
 - [11] D.P. Jackson, R.E. Goldstein, and A.O. Cebers, *Phys. Rev. E* **50**, 298 (1994).
 - [12] I. Drikis, J.C. Bacri, and A. Cebers, *Magnetohydrodynamics* (N.Y.) **35**, 157 (1999).